# GUESSING CLUBS FOR aD, NON D-SPACES

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Abstract. We prove that there exists a 0-dimensional, scattered $T_2$ space $X$ such that $X$ is aD but not linearly D, answering a question of Arhangel'skii. The constructions are based on Shelah's club guessing principles.

## 1. Introduction

The notion of a *D-space* was probably first introduced by van Douwen and since than, many work had been done in this topic. Investigating the properties of *D*-spaces and the connections between other covering properties led to the definition of *aD-spaces*, defined by Arhangel'skii in [2]. As it turned out, property aD is much more docile then property D. In [3] Arhangel'skii asked the following:

**Problem 4.6.** Is there a Tychonoff aD-space which is not a D-space?

A negative answer to this question would settle almost all of the questions about the relationship of classical covering properties to property D. Quite similarly, Guo and Junnila in [6] asked the following about a weakening of property D:

**Problem 2.12.** Is every aD-space linearly D?

In G. Gruenhage's survey on D-spaces [5], another version of this question is stated (besides the original Arhangel'skii), namely:

**Question 3.6(2)** Is every scattered, aD-space a D-space?

The main result of this paper is the following answer to the questions above.

**Theorem 1.1.** *There exists a 0-dimensional $T_2$ space $X$ such that $X$ is scattered, aD and non linearly D.*

In [9] the author showed that the existence of a locally countable, locally compact space $X$ of size $\omega_1$ which is aD and non linearly D is independent of ZFC. Here we refine those methods and using Shelah's club guessing theory we answer the above questions in ZFC.

The paper has the following structure. Sections 2, 3 and 4 gather all the necessary facts about D-spaces, MAD families and club guessing. In Section 5 we define spaces $X[\lambda, \mu, \mathcal{M}, \underline{C}]$, where $\lambda$ and $\mu = cf(\mu)$ are cardinals, $\mathcal{M}$ is a MAD family on $\mu$ and $\underline{C}$ is a guessing sequence. It is shown in Claim 5.2 that

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(0) $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is always $T_2$, 0-dimensional and scattered.

Section 6 contains two important results:

(1) $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is not linearly D if $cf(\lambda) \geq \mu$ (see Corollary 6.3),
(2) $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is aD under certain assumptions (see Corollary 6.9).

Finally in Section 7 we show how to produce such spaces $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ depending on the cardinal arithmetic and using Shelah's club guessing.

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## 2. Definitions

An *open neighborhood assignment* (ONA, in short) on a space $(X, \tau)$ is a map $U : X \to \tau$ such that $x \in U(x)$ for every $x \in X$. X is said to be a *D-space* if for every neighborhood assignment $U$, one can find a closed discrete $D \subseteq X$ such that $X = \bigcup_{d \in D} U(d) = \bigcup U[D]$ (such a set $D$ is called a *kernel for* $U$). In [2] the authors introduced property *aD*:

**Definition 2.1.** *A space $(X, \tau)$ is said to be* aD *iff for each closed $F \subseteq X$ and for each open cover $\mathcal{U}$ of $X$ there is a closed discrete $A \subseteq F$ and $\phi : A \to \mathcal{U}$ with $a \in \phi(a)$ for all $a \in A$ such that $F \subseteq \cup\phi[A]$.*

It is clear that D-spaces are aD. Proving that a space is aD, the notion of an *irreducible space* will play a key role. A space $X$ is *irreducible* iff every open cover $\mathcal{U}$ has a *minimal open refinement* $\mathcal{U}_0$; meaning that no proper subfamily of $\mathcal{U}_0$ covers $X$. In [3] Arhangel'skii showed the following equivalence.

**Theorem 2.2** ([3, Theorem 1.8])**.** *A $T_1$-space $X$ is an aD-space if and only if every closed subspace of $X$ is irreducible.*

Another generalization of property D is due to Guo and Junnila [6]. For a space $X$ a cover $\mathcal{U}$ is *monotone* iff it is linearly ordered by inclusion.

**Definition 2.3.** *A space $(X, \tau)$ is said to be* linearly D *iff for any ONA $U : X \to \tau$ for which $\{U(x) : x \in X\}$ is monotone, one can find a closed discrete set $D \subseteq X$ such that $X = \bigcup U[D]$.*

We will use the following characterization of linear D property. A set $D \subseteq X$ is said to be *$\mathcal{U}$-big* for a cover $\mathcal{U}$ iff there is no $U \in \mathcal{U}$ such that $D \subseteq U$.

**Theorem 2.4** ([6, Theorem 2.2])**.** *The following are equivalent for a $T_1$-space $X$:*

*(1) $X$ is linearly D.*
*(2) For every non-trivial monotone open cover $\mathcal{U}$ of $X$, there exists a closed discrete $\mathcal{U}$-big set in $X$.*

## 3. Notes on MAD families

As MAD families will play an essential part in our constructions we observe some easy facts about them. Let $\mu$ be any infinite cardinal. We call $\mathcal{M} \subseteq [\mu]^\mu$ an *almost disjoint family* if $|M \cap N| < \mu$ for all distinct $M, N \in \mathcal{M}$. $\mathcal{M}$ is a *maximal almost disjoint family* (in short, a *MAD family*) if for all $A \in [\mu]^\mu$ there is some $M \in \mathcal{M}$ such that $|A \cap M| = \mu$.

We will use the following rather trivial combinatorial fact.

**Claim 3.1.** *Let $\mathcal{M} \subseteq [\mu]^\mu$ be a MAD family and $\mathcal{M} = \{M^\varphi : \varphi < \kappa\}$. Suppose that $N \in [\mu]^\mu$ and $|N \setminus \cup\mathcal{M}'| = \mu$ for all $\mathcal{M}' \in [\mathcal{M}]^{<\mu}$. Then $|\Phi| > \mu$ for $\Phi = \{\varphi < \kappa : |N \cap M^\varphi| = \mu\}$.*

*Proof.* If $|\Phi| < \mu$ then with $\widetilde{N} = N \setminus \bigcup\{M^\varphi : \varphi \in \Phi\} \in [\mu]^\mu$ we can extend the MAD family, which is a contradiction. If $|\Phi| = \mu$ then let $\Phi = \{\varphi_\zeta : \zeta < \mu\}$. By transfinite induction, construct $\widetilde{N} = \{n_\xi : \xi < \mu\}$ such that $n_\xi \in N \setminus (\bigcup\{M^{\varphi_\zeta} : \zeta < \xi\} \cup \{n_\zeta : \zeta < \xi\})$ for $\xi < \mu$. It is straightforward that $\widetilde{N} \notin \mathcal{M}$ and $\mathcal{M} \cup \{\widetilde{N}\}$ is almost disjoint, which is a contradiction. □

From our point of view the sizes of MAD families are important. Clearly there is a MAD family on $\omega$ of size $2^\omega$. The analogue of this does not always hold for $\omega_1$. Baumgartner in [4] proves that it is consistent with ZFC that there is no almost disjoint family on $\omega_1$ of size $2^{\omega_1}$. However, we have the following fact.

**Claim 3.2.** *If $2^\omega = \omega_1$ then there is a MAD family $\mathcal{M}$ on $\omega_1$ of size $2^{\omega_1}$.*

In Section 7 we use *nonstationary MAD families* $\mathcal{M}_{NS} \subseteq [\mu]^\mu$ meaning that $\mathcal{M}_{NS}$ is a MAD family such that every $M \in \mathcal{M}_{NS}$ is nonstationary in $\mu$. Observe, that using Zorn's lemma to almost disjoint families of nonstationary sets of $\mu$ we can get nonstationary MAD families.

## 4. Fragments of Shelah's club guessing

The constructions of the upcoming sections will use the following amazing results of Shelah. For a cardinal $\lambda$ and a regular cardinal $\mu$ let $S^\lambda_\mu$ denote the ordinals in $\lambda$ with cofinality $\mu$. For an $S \subseteq S^\lambda_\mu$ an *S-club sequence* is a sequence $\underline{C} = \langle C_\delta : \delta \in S\rangle$ such that $C_\delta \subseteq \delta$ is a club in $\delta$ of order type $\mu$.

**Theorem 4.1** ([7, Claim 2.3])**.** *Let $\lambda$ be a cardinal such that $cf(\lambda) \geq \mu^{++}$ for some regular $\mu$ and let $S \subseteq S^\lambda_\mu$ stationary. Then there is an $S$-club sequence $\underline{C} = \langle C_\delta : \delta \in S\rangle$ such that for every club $E \subseteq \lambda$ there is $\delta \in S$ (equivalently, stationary many) such that $C_\delta \subseteq E$.*

A detailed proof of Theorem 4.1 can be found in [1, Theorem 2.17].

**Theorem 4.2** ([8, Claim 3.5])**.** *Let $\lambda$ be a cardinal such that $\lambda = \mu^+$ for some uncountable, regular $\mu$ and $S \subseteq S^\lambda_\mu$ stationary. Then there is an $S$-club sequence $\underline{C} = \langle C_\delta : \delta \in S\rangle$ such that $C_\delta = \{\alpha^\delta_\zeta : \zeta < \mu\} \subseteq \delta$ and for every club $E \subseteq \lambda$ there is $\delta \in S$ (equivalently, stationary many) such that:*

$$\{\zeta < \mu : \alpha^\delta_{\zeta+1} \in E\} \text{ is stationary.}$$

For a detailed proof, see [10].

## 5. The general construction

**Definition 5.1.** *Let $\lambda > \mu = cf(\mu)$ be infinite cardinals. Let $\mathcal{M} \subseteq [\mu]^\mu$ be a MAD family, $\mathcal{M} = \{M^\varphi : \varphi < \kappa\}$ and let $\underline{C} = \{C_\alpha : \alpha \in S^\lambda_\mu\}$ denote an $S^\lambda_\mu$-club sequence. We define a topological space $X = X[\lambda, \mu, \mathcal{M}, \underline{C}]$ as follows. The underlying set of our topology will be a subset of the product $\lambda \times \kappa$. Let*

- $X_\alpha = \{(\alpha, 0)\}$ *for* $\alpha \in \lambda \setminus S^\lambda_\mu$,
- $X_\alpha = \{\alpha\} \times \kappa$ *for* $\alpha \in S^\lambda_\mu$,
- $X = \bigcup\{X_\alpha : \alpha < \lambda\}$.

*Let* $C_\alpha = \{a^\xi_\alpha : \xi < \mu\}$ *denote the increasing enumeration for* $\alpha \in S^\lambda_\mu$. *For each* $\alpha \in S^\lambda_\mu$ *let*

- $I^\xi_\alpha = (a^\xi_\alpha, a^{\xi+1}_\alpha]$ *for* $\xi \in succ(\mu) \cup \{0\}$,
- $I^\xi_\alpha = [a^\xi_\alpha, a^{\xi+1}_\alpha]$ *for* $\xi \in \lim(\mu)$.

*Note that* $\bigcup\{I^\xi_\alpha : \xi < \mu\} = (a^0_\alpha, \alpha)$ *is a disjoint union.*

*Define the topology on* $X$ *by neighborhood bases as follows;*

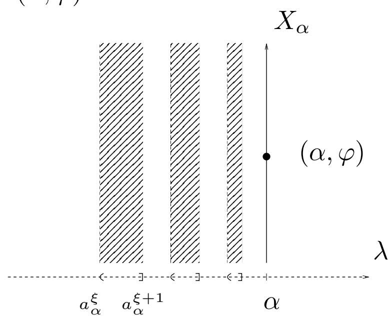
*(i) for* $\alpha \in S^\lambda_\mu$ *and* $\varphi < \kappa$ *let*

$$U((\alpha,\varphi),\eta) = \{(\alpha,\varphi)\} \cup \bigcup\{X_\gamma : \gamma \in \cup\{I^\xi_\alpha : \xi \in M^\varphi \setminus \eta\}\} \text{ for } \eta < \mu$$

*and let*

$$B(\alpha,\varphi) = \{U((\alpha,\varphi),\eta) : \eta < \mu\}$$

*be a base for the point* $(\alpha,\varphi)$.




*(ii) for* $\alpha \in S^\lambda_{<\mu} \cup succ(\lambda) \cup \{0\}$ *let* $(\alpha,0)$ *be an isolated point,*

*(iii) for* $\alpha \in S^\lambda_{\mu'}$ *where* $\mu' > \mu$ *let*

$$U(\alpha,\beta) = \bigcup\{X_\gamma : \beta < \gamma \le \alpha\} \text{ for } \beta < \alpha$$

*and let*

$$B(\alpha) = \{U(\alpha,\beta) : \beta < \alpha\}$$

*be a base for the point* $(\alpha, 0)$.

It is straightforward to check that these *basic open sets* form neighborhood bases.

$\star$

Fix some cardinals $\lambda > \mu = cf(\mu)$, a MAD family $\mathcal{M} = \{M^\varphi : \varphi < \kappa\} \subseteq [\mu]^\mu$ and $S^\lambda_\mu$-club sequence $\underline{C}$. In the following $X = X[\lambda,\mu,\mathcal{M},\underline{C}]$.

**Claim 5.2.** *The space* $X[\lambda,\mu,\mathcal{M},\underline{C}]$ *is 0-dimensional,* $T_2$ *and scattered. Observe that*

*(a)* $X_\alpha$ *is closed discrete for all* $\alpha < \lambda$, *moreover*
*(b)* $\bigcup\{X_\alpha : \alpha \in A\}$ *is closed discrete for all* $A \in [\lambda]^{<\mu}$,
*(c)* $X_{\le\alpha} = \bigcup\{X_\beta : \beta \le \alpha\}$ *is clopen for all* $\alpha < \lambda$.

*Proof.* First we prove that $X[\lambda,\mu,\mathcal{M},\underline{C}]$ is $T_2$. Note that

$(*)$ $\bigcup\{X_\gamma : \beta < \gamma \le \alpha\}$ is clopen for all $\beta < \alpha < \lambda$.

Thus $(\alpha,\varphi), (\alpha',\varphi') \in X$ can be separated trivially if $\alpha \ne \alpha'$. Suppose that $\alpha = \alpha' \in S^\lambda_\mu$ and $\varphi \ne \varphi' < \kappa$. There is $\eta < \mu$ such that $(M^\varphi \cap M^{\varphi'}) \setminus \eta = \emptyset$ since $|M^\varphi \cap M^{\varphi'}| < \mu$. Thus $U((\alpha,\varphi),\eta) \cap U((\alpha,\varphi'),\eta) = \emptyset$.

Next we show that $X[\lambda,\mu,\mathcal{M},\underline{C}]$ is 0-dimensional. By $(*)$ it is enough to prove that $U((\alpha,\varphi),\eta)$ is closed for all $\alpha \in S^\lambda_\mu$, $\varphi < \kappa$ and $\eta < \mu$. Suppose $x = (\alpha',\varphi') \in$

$X \setminus U((\alpha,\varphi),\eta)$, we want to separate $x$ from $U((\alpha,\varphi),\eta)$ by an open set. Let $\alpha = \alpha'$. There is $\eta' < \mu$ such that $(M^\varphi \cap M^{\varphi'}) \setminus \eta' = \emptyset$, thus $U((\alpha,\varphi),\eta) \cap U((\alpha,\varphi'),\eta') = \emptyset$. Let $\alpha \neq \alpha'$. If $\alpha' \in S^\lambda_{<\mu} \cup \mathrm{succ}(\lambda) \cup \{0\}$ then $x$ is isolated, thus we are done. Suppose $\alpha \in S^\lambda_{\mu'}$ where $\mu' \geq \mu$. Then $\beta = \sup(C_\alpha \setminus \alpha') < \alpha'$ thus $U(\alpha',\beta) \cap U((\alpha,\varphi),\eta) = \emptyset$.

$X[\lambda,\mu,\mathcal{M},\underline{C}]$ is scattered since $X[\lambda,\mu,\mathcal{M},\underline{C}]$ is right separated by the lexicographical ordering on $\lambda \times \kappa$.

(a) and (c) is trivial, we prove (b). Suppose $x = (\alpha',\varphi') \in X$, we prove that there is a neighborhood $U$ of $x$ such that $|U \cap \bigcup\{X_\alpha : \alpha \in A\}| \leq 1$. If $\alpha' \in S^\lambda_{<\mu} \cup \mathrm{succ}(\lambda) \cup \{0\}$ then $x$ is isolated, thus we are done. Suppose $\alpha \in S^\lambda_{\mu'}$ where $\mu' \geq \mu$. Then $\beta = \sup(A \setminus \alpha') < \alpha'$ thus the open set $U = \{x\} \cup \bigcup\{X_\gamma : \beta < \gamma < \alpha\}$ will do the job. $\square$

## 6. Focusing on property D and aD

Again fix some cardinals $\lambda > \mu = cf(\mu)$, a MAD family $\mathcal{M} = \{M^\varphi : \varphi < \kappa\} \subseteq [\mu]^\mu$ and $S^\lambda_\mu$-club sequence $\underline{C}$. Our next aim is to investigate the spaces $X = X[\lambda,\mu,\mathcal{M},\underline{C}]$ concerning property D and aD.

**Definition 6.1.** *Let $\pi(F) = \{\alpha < \lambda : F \cap X_\alpha \neq \emptyset\}$ for $F \subseteq X$. $F$ is said to be* (un)bounded *if $\pi(F)$ is (un)bounded in $\lambda$.*

**Claim 6.2.** *If $F \subseteq X$ and $\pi(F)$ accumulates to $\alpha \in S^\lambda_\eta$ such that $\mu \leq \eta < \lambda$ then $F' \cap X_\alpha \neq \emptyset$.*

*Proof.* If $\eta > \mu$ then $X_\alpha = \{(\alpha,0)\}$ and each neighborhood $U(\alpha,\beta)$ of $(\alpha,0)$ intersects $F$. Thus $F' \cap X_\alpha \neq \emptyset$. Let us suppose that $\pi(F)$ accumulates to $\alpha \in S^\lambda_\mu$. Since $\bigcup\{I^\xi_\alpha : \xi < \mu\} = (a^0_\alpha, \alpha)$, the set $N = \{\xi < \mu : I^\xi_\alpha \cap \pi(F) \neq \emptyset\}$ has cardinality $\mu$. Thus there is some $\varphi < \kappa$ such that $|N \cap M^\varphi| = \mu$, since $\mathcal{M}$ is MAD family. It is straightforward that $(\alpha,\varphi) \in F'$ since $U((\alpha,\varphi),\eta) \cap F \neq \emptyset$ for all $\eta < \mu$. $\square$

**Corollary 6.3.** *If $cf(\lambda) \geq \mu$ then a closed unbounded subspace $F \subseteq X$ is not a linearly D-subspace of $X$. Hence $X[\lambda,\mu,\mathcal{M},\underline{C}]$ is not a linearly D-space.*

*Proof.* Let $F \subseteq X$ be closed unbounded. $|\pi(D)| < \mu$ for every closed discrete $D \subseteq X$ by Claim 6.2. Thus there is no big closed discrete set for the open cover $\{X_{\leq\alpha} : \alpha < \lambda\}$ which shows that $F$ is not linearly D by Theorem 2.4. $\square$

Our aim now is to prove that in certain cases the space $X[\lambda,\mu,\mathcal{M},\underline{C}]$ is an aD-space, equivalently every closed subspace of it is irreducible.

**Claim 6.4.** *Every closed, bounded subspace $F \subseteq X$ is a D-subspace of $X$; hence $F$ is irreducible.*

*Proof.* We prove that $F \subseteq X$ is a D-subspace of $X$ by induction on $\alpha = \sup \pi(F) < \lambda$. Let $U : F \to \tau$ be an ONA. If $\alpha$ is a successor (or $\alpha = 0$), then $F_0 = F \setminus U((\alpha,0))$ is closed and $\sup(F_0) < \alpha$ thus we are easily done by induction.

Let $\alpha \in S^\lambda_{\mu'}$ where $\mu \leq \mu' < \lambda$. Then $\sup \pi(F_0) < \alpha$ where $F_0 = F \setminus \cup U[X_\alpha \cap F]$ by Claim 6.2. Thus we are easily done by induction and the fact that $X_\alpha$ is closed discrete.

Now let $\nu = cf(\alpha) < \mu$, let $\sup\{\alpha_\xi : \xi < \nu\} = \alpha$ such that $\alpha_0 = 0$ and $\{\alpha_\xi : \xi < \nu\}$ is strictly increasing. Let $J_\xi = \bigcup\{X_\gamma : \alpha_\xi \leq \gamma \leq \alpha_{\xi+1}\}$ if $\xi < \nu$ is limit or $\xi = 0$ and $J_\xi = \bigcup\{X_\gamma : \alpha_\xi < \gamma \leq \alpha_{\xi+1}\}$ if $\xi < \nu$ is a successor. Let $J_\nu = X_\alpha$. Clearly $\{J_\xi : \xi \leq \nu\}$ is a discrete family of disjoint clopen sets such that

$\bigcup\{J_\xi : \xi \le \nu\} = X_{\le\alpha}$. $F = \bigcup\{F^\xi : \xi \le \nu\}$ where $F^\xi = F \cap J_\xi$ is closed for $\xi \le \nu$. By induction, for all $\xi < \nu$ there is some closed discrete kernel $D^\xi \subseteq F^\xi$ for the restriction of $U$ to $F^\xi$. Let $D^\nu = F^\nu$. Then $D = \bigcup\{D^\xi : \xi \le \nu\}$ is closed discrete and $F \subseteq \cup U[D]$. $\square$

To handle the unbounded closed subsets we need the following definition.

**Definition 6.5.** *Let $F_\alpha = F \cap X_\alpha$ for $F \subseteq X$ and $\alpha < \lambda$. A* subset $F \subseteq X$ is high enough *if*

$$|\{\alpha < \lambda : |F_\alpha| = |F|\}| \ge \mu.$$

*We say that a* subset $F \subseteq X$ is high *if every closed unbounded subset of $F$ is high enough.*

The following rather technical claim will be useful.

**Claim 6.6.** *For any $F \subseteq X$ and ONA $U : F \to \tau$ such that $U(x)$ is a basic open neighborhood of $x \in F$, let*

$$Y_F = \{x \in F : \exists\alpha < \lambda : F_\alpha \subseteq U(x), |F_\alpha| = |F|\},$$
$$\Gamma_F = \{\alpha < \lambda : |F_\alpha| = |F|, \exists x \in F : F_\alpha \subseteq U(x)\}.$$

*If $F$ is closed and high enough then $Y_F, \Gamma_F \ne \emptyset$.*

*Proof.* Since $Y_F \ne \emptyset$ iff $\Gamma_F \ne \emptyset$, it is enough to show that there is some $x \in Y_F$. Since $F$ is high enough, $|Z| \ge \mu$ for $Z = \{\alpha' < \lambda : |F| = |F_{\alpha'}|\}$. Let $D = \bigcup\{F_{\alpha'} : \alpha' \in Z\} \subseteq F$. Let $\beta \in S^\lambda_\mu$ be an accumulation point of $Z = \pi(D)$. Then by Claim 6.2 there is some $x \in D' \cap X_\beta$ thus $x \in F$. Clearly $x \in Y_F$. $\square$

**Theorem 6.7.** *If the closed unbounded $F \subseteq X$ is high then $F$ is irreducible.*

*Proof.* Suppose that $\mathcal{U}$ is an open cover of $F$. We can suppose that we refined it to the form $\{U(x) : x \in F\}$ where each $U(x)$ is basic open. From Claim 6.6 we know that $Y_F, \Gamma_F \ne \emptyset$. We define $Y^\xi \subseteq F$ by induction.

- Let $\alpha_0 \in \Gamma_F$ and $Y^0 = \{x \in Y_F : F_{\alpha_0} \subseteq U(x)\}$. Fix some $h^0 : Y^0 \to F_{\alpha_0}$ injection; this exists because $|F_{\alpha_0}| = |F| \ge |Y_F| \ge |Y^0|$.
- Suppose we defined $\alpha_\zeta < \lambda$ and $Y^\zeta$ for $\zeta < \xi$. Let
  $$F^\xi = F \setminus \left(\bigcup\left\{U(x) : x \in \cup\{Y^\zeta : \zeta < \xi\}\right\} \cup X_{\le\alpha}\right)$$
  where $\alpha = \sup\{\alpha_\zeta : \zeta < \xi\}$.
- If $F^\xi$ is bounded then stop. Notice that $F_\xi$ is bounded iff $F \setminus \bigcup\left\{U(x) : x \in \cup\{Y^\zeta : \zeta < \xi\}\right\}$ is bounded.
- Suppose $F^\xi$ is unbounded. $F^\xi \subseteq F$ is closed either thus $F^\xi$ is high enough since $F$ is high. Hence $Y_{F^\xi}, \Gamma_{F^\xi} \ne \emptyset$.
- Let $\alpha_\xi \in \Gamma_{F^\xi}$; thus $|F^\xi_{\alpha_\xi}| = |F^\xi|$ and $F^\xi_{\alpha_\xi}$ is covered by some $U(x)$ for $x \in F^\xi$. Let $Y^\xi = \{x \in Y_{F^\xi} : F^\xi_{\alpha_\xi} \subseteq U(x)\}$. Fix some $h^\xi : Y^\xi \to F^\xi_{\alpha_\xi}$ injection; this exists because $|F^\xi_{\alpha_\xi}| = |F^\xi| \ge |Y_{F^\xi}| \ge |Y^\xi|$.

**Lemma 6.8.** *The induction stops before $\mu$ many steps.*

*Proof.* Suppose we defined this way $\{\alpha_\xi : \xi < \mu\}$ and let $\alpha = \sup\{\alpha_\xi : \xi < \mu\} \in S^\lambda_\mu$. Let $D = \bigcup\{F_{\alpha_\xi} : \xi < \mu\}$. By Claim 6.2 there is some $x \in D' \cap X_\alpha$, thus $x \in F$ either. Clearly $F_{\alpha_\xi} \subseteq U(x)$ for $\mu$ many $\xi < \mu$. By the definition of the induction

(∗) for every $\zeta < \xi < \mu$ and every $y \in Y^\zeta$: $F^\xi_{\alpha_\xi} \cap U(y) = \emptyset$

Clearly by $(*)$, $x \notin Y^\zeta$ for all $\zeta < \mu$ since there is $\zeta < \xi < \mu$ such that $F^\xi_{\alpha_\xi} \subseteq U(x)$. Moreover $x \notin U(y)$ for every $y \in Y^\zeta$ and $\zeta < \mu$; if $x \in U(y)$ then since $x \neq y$ there is some $\beta < \alpha$ such that $\bigcup\{X_\gamma : \beta < \gamma \leq \alpha\} \subseteq U(y)$. This contradicts $(*)$ since there is $\zeta < \xi < \mu$ such that $\beta < \alpha_\xi$, thus $F^\xi_{\alpha_\xi} \subseteq U(y)$. Thus $x \in F^\xi$ for all $\xi < \mu$. Then $x \in Y^\xi$ for all $\xi < \mu$ such that $F_{\alpha_\xi} \subseteq U(x)$. This is a contradiction. □

Thus let us suppose that the induction stopped at step $\xi < \mu$, meaning that $\widetilde{F} = F \setminus \bigcup\{U(x) : x \in Y\}$ is bounded where $Y = \cup\{Y^\zeta : \zeta < \xi\}$. Let $h = \bigcup\{h^\zeta : \zeta < \xi\}$, $h : Y \to F$ is a 1-1 function since the sets $\mathrm{dom}(h^\zeta) = Y^\zeta$ and $ran(h^\zeta) \subseteq F^\zeta_{\alpha_\zeta}$ are pairwise disjoint for $\zeta < \xi$. Note that $\mathrm{ran}(h) \subseteq \bigcup\{F_{\alpha_\zeta} : \zeta < \xi\}$ is closed discrete by Claim 5.2. For $x \in Y$ let

$$U_0(x) = (U(x) \setminus \mathrm{ran}(h)) \cup \{h(x)\},$$

note that $U_0(x)$ is open. Then

$$\bigcup\{U_0(x) : x \in Y\} = \bigcup\{U(x) : x \in Y\}$$

is a minimal open refinement, since $h(x)$ is only covered by $U_0(x)$ for all $x \in Y$. Let $\mathcal{U}_0 = \{U_0(x) : x \in Y\}$

Let $V(x) = U(x) \setminus \bigcup\{F_{\alpha_\zeta} : \zeta < \xi\}$. Then $\mathcal{V} = \{V(x) : x \in \widetilde{F}\}$ is an open cover of $\widetilde{F}$, refining $\mathcal{U}$; $F_{\alpha_\zeta} \cap \widetilde{F} = \emptyset$ by construction for all $\zeta < \xi$. $\widetilde{F}$ is closed and bounded thus irreducible by Claim 6.4, hence there is an irreducible open refinement $\mathcal{V}_0$ of $\mathcal{V}$. It is straightforward that $\mathcal{V}_0 \cup \mathcal{U}_0$ is a minimal open refinement of $\mathcal{U}$ covering $F$. □

**Corollary 6.9.** *Suppose that $\lambda > \mu = cf(\mu)$ are infinite cardinals such that $cf(\lambda) \geq \mu$. Let $\mathcal{M} = \{M^\varphi : \varphi < \kappa\} \subseteq [\mu]^\mu$ be a MAD family and $\underline{C}$ an $S^\lambda_\mu$-club sequence. If $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is high then $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is a 0-dimensional, Hausdorff, scattered space which is aD however not linearly D.*

*Proof.* $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is 0-dimensional, Hausdorff and scattered by Claim 5.2 and not linearly D by Corollary 6.3. It suffices to show that every closed $F \subseteq X$ is irreducible. If $F$ is bounded then $F$ is a D-space by Claim 6.4 hence irreducible. If $F$ is unbounded, then $F$ is high since $X$ is high. Hence $F$ is irreducible by Theorem 6.7. □

## 7. Examples of aD, non linearly D-spaces

In this section we give examples of aD, non linearly D-spaces of the form $X = X[\lambda, \mu, \mathcal{M}, \underline{C}]$. First let us make an observation.

**Claim 7.1.** *If $C_\alpha \subseteq \pi(F)'$ for a closed $F \subseteq X$ and $\alpha \in S^\lambda_\mu$, then $F_\alpha = X_\alpha$.*

*Proof.* Clearly $\bigcup\{X_\gamma : \gamma \in I^\xi_\alpha\} \cap F \neq \emptyset$ for all $\xi < \mu$. Thus every point in $X_\alpha$ is an accumulation point of $F$, thus $F_\alpha = X_\alpha$ since $F$ is closed. □

Corollaries 7.3 and 7.5 below give certain examples of high $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ spaces.

**Proposition 7.2.** *Suppose that $\mu$ is a regular cardinal, $cf(\lambda) \geq \mu^{++}$. Let $\underline{C}$ be an $S^\lambda_\mu$-club guessing sequence from Theorem 4.1. If $\mathcal{M} \subseteq [\mu]^\mu$ is a MAD family of size at least $\lambda$ then $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is high.*

*Proof.* Let $F \subseteq X$ closed, unbounded. Then $\pi(F)'$ is a club in $\lambda$, hence there exists a stationary $S \subseteq S^\lambda_\mu$ such that $C_\alpha \subseteq \pi(F)'$ for all $\alpha \in S$. Thus $F_\alpha = X_\alpha$ by Claim 7.1 hence $|F_\alpha| = |\mathcal{M}| = |X|$ for all $\alpha \in S$. $\square$

**Corollary 7.3.** *(1) Suppose that $2^\omega \geq \omega_2$. Let $\mathcal{M}$ be a MAD family on $\omega$ of size $2^\omega$ and let $\underline{C}$ be an $S^{\omega_2}_\omega$-club guessing sequence from Theorem 4.1. Then $X[\omega_2, \omega, \mathcal{M}, \underline{C}]$ is high.*

*(2) Suppose that $2^\omega = \omega_1$ and $2^{\omega_1} \geq \omega_3$. Let $\mathcal{M}$ be a MAD family on $\omega_1$ of size $2^{\omega_1}$ (exists by Claim 3.2) and let $\underline{C}$ be an $S^{\omega_3}_{\omega_1}$-club guessing sequence from Theorem 4.1. Then $X[\omega_3, \omega_1, \mathcal{M}, \underline{C}]$ is high.*

**Proposition 7.4.** *Suppose that $\lambda = \mu^+ > \mu = cf(\mu) > \omega$ and let $\underline{C}$ be an $S^{\mu^+}_\mu$-club guessing sequence from Theorem 4.2. If there is a nonstationary MAD family $\mathcal{M}_{NS} \subseteq [\mu]^\mu$ such that $|\mathcal{M}_{NS}| = \mu^+$ then $X = X[\mu^+, \mu, \mathcal{M}_{NS}, \underline{C}]$ is high.*

*Proof.* Let $\mathcal{M}_{NS} = \{M^\varphi : \varphi < \mu^+\}$ and $\underline{C} = \langle C_\alpha : \alpha \in S^{\mu^+}_\mu \rangle$ such that $C_\alpha = \{a^\xi_\alpha : \xi < \mu\} \subseteq \alpha$. Suppose that the closed $F \subseteq X$ is unbounded. Then $\pi(F)'$ is a club in $\mu^+$, hence there exists a stationary $S \subseteq S^{\mu^+}_\mu$ such that

$$N_\alpha = \{\xi < \mu : a^{\xi+1}_\alpha \in \pi(F)'\} \text{ is stationary in } \mu$$

for all $\alpha \in S$. Fix any $\alpha \in S$, we prove that $|F_\alpha| = |F|$. $N_\alpha$ is stationary so by applying Claim 3.1 we get that $|\Phi_\alpha| = \mu^+$ for $\Phi_\alpha = \{\varphi < \mu^+ : |N_\alpha \cap M^\varphi| = \mu\}$. Note that $F \cap \bigcup\{X_\gamma : \gamma \in I^\xi_\alpha\} \neq \emptyset$ for $\xi \in N_\alpha$. Thus $(\alpha, \varphi)$ is an accumulation point of $F$ for $\varphi \in \Phi_\alpha$, hence $\{\alpha\} \times \Phi_\alpha \subseteq F_\alpha$. Thus $|F_\alpha| = \mu^+ = |X|$. $\square$

**Corollary 7.5.** *Suppose that $2^{\omega_1} = \omega_2$. Let $\underline{C}$ be an $S^{\omega_2}_{\omega_1}$-club guessing sequence from Theorem 4.2 and let $\mathcal{M}_{NS}$ be a nonstationary MAD family on $\omega_1$. Then $X[\omega_2, \omega_1, \mathcal{M}_{NS}, \underline{C}]$ is high.*

Thus, by all means we can deduce the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Note that in any model of ZFC, either $(2^\omega \geq \omega_2)$ or $(2^\omega = \omega_1 \wedge 2^{\omega_1} \geq \omega_3)$ or $(2^{\omega_1} = \omega_2)$. Using Corollaries 7.3 and 7.5 above, depending on the sizes of $2^\omega$ and $2^{\omega_1}$, we see that there exists a high $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ space. We are done by Corollary 6.9. $\square$

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